Assignment 3

~ Solutions ~

25 points

① 5 points Define propositional variables:

1p p: "4 does not divide n"

q: "4 divides $n^3-6n^2+11n-6$"

The theorem to be proved is of type $p \to q$.

Note that $p = p_1 \vee p_2 \vee p_3$, where

1p p_1: "$n = 4k+1$, for some $k \in \mathbb{Z}$"

p_2: "$n = 4k+2$, for some $k \in \mathbb{Z}$"

p_3: "$n = 4k+3$, for some $k \in \mathbb{Z}$"

Hence, we prove that $p_1 \to q$, $p_2 \to q$ and $p_3 \to q$.

Case 1. $n = 4k+1$, for some $k \in \mathbb{Z}$. Then

1p
$$n^3-6n^2+11n-6 = (4k+1)^3 - 6(4k+1)^2 + 11(4k+1) - 6$$
$$= 4(16k^3-12k^2+2k) = 4m_1, \text{ with } m_1 = 16k^3-12k^2+2k$$

Since $m_1 \in \mathbb{Z}$ and $n = 4m_1$, we conclude that q is true

Case 2. $n = 4k+2$, for some $k \in \mathbb{Z}$. Then

1p
$$n^3-6n^2+11n-6 = (n-1)(n-2)(n-3) = (4k+1)\cdot 4k \cdot (4k-1)$$
$$= 4 \cdot m_2, \text{ with } m_2 = k(4k+1)(4k-1)$$

Since $m_2 \in \mathbb{Z}$ and $n = 4m_2$, we conclude that q is true

Case 3. $n = 4k+3$, for some $k \in \mathbb{Z}$. Then

1p
$$n^3-6n^2+11n-6 = (n-1)(n-2)(n-3) = (4k+2)(4k+1)\cdot 4k = 4m_3,$$

with $m_3 = k(4k+2)(4k+1)$. Since $m_3 \in \mathbb{Z}$ and $n = 4m_3$, we conclude that q is true.

② (a) $(B \cap C) \cup (B \cap \bar{A}) = B \cap (C \cup \bar{A})$ $\boxed{\text{4 points}}$

1p $= B \cap \overline{\overline{(C \cup \bar{A})}}$

$= B \setminus \overline{(C \cup \bar{A})}$

(b) i. The statement is not true in general.

Counterexample: Let $\mathcal{U} = \{1, 2, 3, 4, 5\}$

1p $A = \{1, 2, 3\}$

$B = \{1, 2\}$

$C = \{1, 2, 4\}$

$A - B = \{3\}$

$A - C = \{3\}$ so $A - B = A - C$. However, $B \neq C$.

ii. The statement is true. To show $A \cap B = A \cap C$, we show

~~Let~~ I. $A \cap B \subseteq A \cap C$ and

1p II. $A \cap B \supseteq A \cap C$

I. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. It follows that $x \notin A - B$ and so $x \notin A - C$.

Since $x \notin A - C$ and $x \in A$, we must have $x \in C$ and therefore, $x \in A \cap C$

1p

II. Follows similarly (i.e ~~the~~ show that $x \in A \cap C$ implies $x \in A \cap B$ by following the proof at I. and replace B by C).

③ (i) (a) Let $(x_1, y_1) \in \mathbb{Z} \times \mathbb{Z}$ and $(x_2, y_2) \in \mathbb{Z} \times \mathbb{Z}$ such that

$\boxed{\text{6 points}}$ $f(x_1, y_1) = f(x_2, y_2)$.

This implies $x_1 - 2y_1 = x_2 - 2y_2$ which does not necessarily imply that $x_1 = x_2$ and $y_1 = y_2$.

1p Hence, f is not one-to-one.

Counterexample: let $x_1 = 2$ and $y_1 = 0$. Then $f(x_1, y_1) = 2$

$x_2 = 4$ and $y_1 = 1$. Then $f(x_2, y_2) = 2$

Therefore $f(x_1, y_1) = f(x_2, y_2)$, but $(x_1, y_1) \neq (x_2, y_2)$

(b) f is onto.

Let $m \in \mathbb{Z}$ such that $f(x,y) = m$. Then $x - 2y = m$.

One solution is to take $x = m$ and $y = 0$.

2p We have $(x,y) = (m,0) \in \mathbb{Z}\times\mathbb{Z}$ and $f(x,y) = m$.

Hence, we proved that, for any $m \in \mathbb{Z}$, there exists a pair (namely $(m,0) \in \mathbb{Z}\times\mathbb{Z}$) such that

$$f(m,0) = m.$$

(ii) Since g is one-to-one and onto, g is bijective.

Hence, g has an inverse.

1p
$$g^{-1} : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}, \quad \text{with } (g\circ g^{-1})_{(x)} = (g^{-1}\circ g)_{(x)} = x.$$

We find the inverse function g^{-1}.

Let $(s,t) \in \mathbb{Z}\times\mathbb{Z}$ such that $g(x,y) = (s,t)$.

Then, $(y, y-x) = (s,t)$, or $\begin{cases} y = s \\ y - x = t \end{cases} \Rightarrow \begin{cases} y = s \\ x = s - t \end{cases}$

2p

Therefore, $g^{-1} : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, $g^{-1}(s,t) = (s-t, s)$

$\boxed{\text{4 points}}$

4 • R is reflexive if
$(a,a) \in R$
$(b,b) \in R$
$(c,c) \in R$ } all true
$(d,d) \in R \to$ not true.

1p Therefore, R is <u>not</u> reflexive

• R is symmetric if $(y,x) \in R$ whenever $(x,y) \in R$

1p R is <u>not</u> symmetric. Counterexample: we have $(b,a) \in R$, but $(a,b) \notin R$

• R is antisymmetric since for all $x, y \in \{a,b,c,d\}$,

1p $x \neq y \to ((x,y) \notin R$ or $(y,x) \notin R)$

(i.e the contrapositive of $((x,y) \in R$ and $(y,x) \in R) \to x = y$

1p • R is transitive. One can verify that

$((x,y) \in R$ and $(y,z) \in R) \to (x,z) \in R$

5 (a) R is reflexive since $(x_1,y_1)R(x_1,y_1)$: $x_1^2-y_1=x_1^2-y_1$

$\boxed{\text{6 points}}$ R is symmetric since

$$(x_1,y_1)R(x_2,y_2) \rightarrow (x_2,y_2)R(x_1,y_1)$$

3p → $$x_1^2-y_1=x_2^2-y_1 \qquad x_2^2-y_2=x_1^2-y_1$$

R is transitive since

$$\left((x_1,y_1)R(x_2,y_2) \text{ and } (x_2,y_2)R(x_3,y_3)\right) \rightarrow (x_1,y_1)R(x_3,y_3)$$

$$\left(x_1^2-y_1=x_2^2-y_2 \text{ and } x_2^2-y_2=x_3^2-y_3\right) \rightarrow x_1^2-y_1=x_3^3-y_3$$

Hence, R is ~~an~~ equivalence relation

(b) A partition of the set A is:

$A_1=\{(-2,3),\ (1,0),\ (\sqrt{3},2),\ (2,3)\}$

$A_2=\{(-\sqrt{3},4),\ (0,1)\}$

2p → $A_3=\{(0,0),\ (1,1)\}$

$A_4=\{(\sqrt{3},-4),\ (\sqrt{6},-1)\}$

A_1, A_2, A_3, A_4 are equivalence classes of R on the set A.

(c)

$$[(-1,1)]_R=\{(x,y)\in\mathbb{R}\times\mathbb{R},\ (-1)^2-1=x^2-y\}$$
$$=\{(x,y)\in\mathbb{R}\times\mathbb{R},\ y=x^2\}$$

1p → This set consists of points in the plane that are located on the parabola with vertex $(0,0)$, opening upwards (i.e the points on the graph of the function $f(x)=x^2$)